Simulation of nonlinear wave eqn in GPU

Bjørn Angelsen, Professor of Medical Imaging, Medical Faculty, NTNU

Content

1. Wave equation in nonlinear, heterogeneous tissue

1.1 Wave equation in the momentum potential

Ultrasound vibrations are conveniently described by Lagrange coordinates (also called Material coordinates) to avoid the nonlinear convective acceleration term found with Euler coordinates. The vetor $\underline{r} = x_i \underline{e}_i$ defines in the Lagrange coordinates the material point at equilibrium, where during a vibration the material point gets the spatial location

$$\underline{\xi}(\underline{r},t) = \underline{r} + \underline{\psi}(\underline{r},t) \tag{1.1}$$

where $\underline{\psi}(\underline{r},t)$ is the displacement as a function of time t for the particle with the equilibrium position $\underline{r}$. The Newton acceleration equation then takes the form [11]

$$\rho \frac{\partial^2 \underline{\psi}}{\partial t^2} = -cof\left(\frac{\partial \underline{\xi}}{\partial \underline{r}}\right)\frac{\partial p}{\partial \underline{r}} \tag{1.2}$$

where $p(\underline{r},t)$ is the pressure in the material and we have defined the matrix

$$cof\left(\frac{\partial \underline{\xi}}{\partial \underline{r}}\right) = Det\left(\frac{\partial \underline{\xi}}{\partial \underline{r}}\right)\left(\frac{\partial \underline{\xi}}{\partial \underline{r}}\right)^{-1} \tag{1.3}$$

$$\frac{\partial \underline{\xi}}{\partial \underline{r}} = \left\{\frac{\partial \xi_i}{\partial x_j}\right\} = \left\{\delta_{ij} + \frac{\partial \psi_i}{\partial x_j}\right\}$$

where δ_{ij} is the Kronecker delta. The relative volume compression in an ultrasound wave is $\delta V/\Delta V = -\nabla \underline{\psi}(\underline{r},t) \sim 10^{-3}$ which allows us to take care of only 1st order terms in the cof calculation. This gives the approximation [11]

$$\rho \frac{\partial^2 \underline{\psi}}{\partial t^2} = -\nabla p\left(1 + \nabla \underline{\psi}\right) + \left(\nabla p \cdot \nabla\right)\underline{\psi} \tag{1.4}$$

We hence see that as we with Lagrange coordinates avoid the nonlinear convection acceleration of the Euler coordinates in (1.2), the force term on the right side is a nonlinear mixture of gradients in the pressure and displacement components. However, for plane waves the right hand side reduces to $-\nabla p$. This is also a good approximation for a focused ultrasound beam where the radius of curvature of the wave fronts, F, is large compared to the wave length λ, i.e. $\lambda / F << 1$. We shall in the following approximate

$$\rho \frac{\partial^2 \underline{\psi}}{\partial t^2} = -\nabla p \tag{1.5}$$

The constitutive material equation for an isotropic material with nonlinear elasticity can be written as

$$\frac{\delta V}{\Delta V} = -\nabla \underline{\psi}(\underline{r},t) = K\left(p(\underline{r},t)\right) + h_p \underset{t}{\otimes} \kappa p(\underline{r},t) \qquad \kappa = \left.\frac{\partial K}{\partial p}\right|_{p=0} \tag{1.6}$$

where *K(p)* is a nonlinear elastic compressibility function for isentropic (very rapid) compression of the material and κ is the linear bulk compressibility of the material. The convolution term represents a dynamic modification when the material compression is not fully isentropic and the term hence introduces conversion of acoustic vibration energy to thermal vibrations, i.e. absorption of acoustic energy to heat. The elasticity function is conveniently split into a linear, dominant component for small pressure amplitudes and a nonlinear component as

$$K(p) = \kappa p - K_n(p) \tag{1.7}$$

For fluids and soft tissue (which has elasticity close to water) it is adequate to use a 2nd order approximation to *K(...)* as

$$K(p) = (1 - \beta_n \kappa p)\kappa p \qquad \beta_n = 1 + B/2A \tag{1.8}$$

where β_n is a nonlinearity parameter defined by the common 1st and 2nd order elastic parameters A and B. For soft tissues we have $\kappa \sim 400\ 10^{-12}\ \mathrm{Pa}^{-1}$ and $\beta_n \sim 5$. For p = 2.5 MPa the linear approximation gives $\delta V/\Delta V = -\nabla \underline{\psi}(\underline{r},t) \sim \kappa p \sim 10^{-3}$. The relative nonlinear term is $K_n(p)/\kappa p = \beta_n \kappa p \sim 5\ 10^{-3}$. For gases the elasticity is stronger and it is useful to maintain the full nonlinear description *K(p)*. For solids with tight atomic bonds the relative volume compression is so low that nonlinear elasticity can generally be neglected for the pressure levels found with medical imaging and non-destructive testing.
For compression waves with zero shear strain one can conveniently introduce the impulse potential

$$\begin{aligned} &\rho \underline{u}(\underline{r},t) = -\nabla \phi(\underline{r},t) & & \underline{u}(\underline{r},t) = \frac{\partial \underline{\psi}(\underline{r},t)}{\partial t} & & \text{a)} \\ &p(\underline{r},t) = \frac{\partial \phi(\underline{r},t)}{\partial t} & \Rightarrow \quad & \phi(\underline{r},t) = \int_0^t d\tau\, p(\underline{r},\tau) & & \text{b)} \end{aligned} \tag{1.9}$$

where *u(r,t)* is the particle vibration velocity. (1.9b) follows from (1.9a) combined with (1.5) and is hence an approximation under the requirement that $\lambda/F \ll 1$. Combining (1.9a) and (1.6) gives

$$\nabla\left(\frac{1}{\rho}\nabla \phi(\underline{r},t)\right) - K'\left(p(\underline{r},t)\right)\frac{\partial^2 \phi(\underline{r},t)}{\partial t^2} - h_p \underset{t}{\otimes} \kappa \frac{\partial^2 \phi(\underline{r},t)}{\partial t^2} = 0 \tag{1.10}$$

Spatial heterogeneity is introduced because soft tissues are composed of different material types (fat, muscle, parenchyma, connective tissue) which gives a spatial variation of the mass density and the elasticity. For the heterogeneous material we separate the material parameters into a slowly varying (scale ~ λ) component and a rapidly varying component, i.e.

$$\rho(\underline{r}) = \rho_a(\underline{r}) + \rho_f(\underline{r}) \qquad K(\underline{r};p) = K_a(\underline{r};p) + K_f(\underline{r};p)$$
$$K_a(\underline{r};p) = \kappa_a(\underline{r})p - K_{na}(\underline{r};p) \quad K_f(\underline{r};p) = \kappa_f(\underline{r})p - K_{nf}(\underline{r};p) \tag{1.11}$$

Note that $1/\rho = 1/\rho_a - \gamma/\rho_a$, $\gamma = \rho_f/\rho_a$, Inserting this into (1.10) and approximating $\nabla\rho_a \approx 0$

$$\underbrace{\nabla^2\phi(\underline{r},t) - \frac{1}{c^2(\underline{r};p)}\frac{\partial^2\phi(\underline{r},t)}{\partial t^2}}_{Nonlinear\ propagation} - \underbrace{h_p \underset{t}{\otimes} \frac{1}{c_0^2}\frac{\partial^2\phi(\underline{r},t)}{\partial t^2}}_{absorption}$$
$$= \underbrace{\frac{\sigma_l(\underline{r})}{c_0^2(\underline{r})}\frac{\partial^2\phi(\underline{r},t)}{\partial t^2} + \nabla\left(\gamma(\underline{r})\nabla\phi(\underline{r},t)\right)}_{Linear\ scattering\ source\ terms} - \underbrace{\frac{\sigma_n(p;\underline{r})}{c_0^2(\underline{r})}\frac{\partial^2\phi(\underline{r},t)}{\partial t^2}}_{\substack{Nonlinear\ scattering\\ source\ term}} \tag{1.12}$$

$$c^2(\underline{r};p) = \frac{1}{\rho_a(\underline{r})K_a'(\underline{r};p)} \qquad c_0^2(\underline{r}) = \frac{1}{\rho_a(\underline{r})\kappa_a(\underline{r})}$$

$$\gamma(\underline{r}) = \frac{\rho_f(\underline{r})}{\rho(\underline{r})} \qquad \sigma_l(\underline{r}) = \frac{\kappa_f(\underline{r})}{\kappa_a(\underline{r})} \qquad \sigma_n(p;\underline{r}) = \frac{K_{nf}'(\underline{r};p)}{\kappa_a(\underline{r})}$$

The left side represents the propagation of the wave, while the rapid spatial fluctuation of the coefficients of the right side terms are scattering sources. The pressure dependency of the wave velocity *c(r;p)* implies that the high pressure in the oscillation propagates with a higher velocity than the low pressure. This produces a nonlinear propagation distortion of the wave oscillation that accumulates with propagation distance and introduces harmonic frequency components in the propagating pulse. Scattering sources including the absorption term have low magnitude and are neglected.

Solution of the nonlinear equation (1.12) must be done through numerical simulations, which we return to in Section III. However, the amplitude of the scattered wave is so low that one can for the scattered wave neglect the nonlinear terms in both the propagation velocity and the scattering sources. The left side propagation operator therefore approximates to a linear operator for the scattered wave, and one can use the Helmholtz-Kirchoff's theorem that transforms (1.12) into an integral equation

$$\phi(\underline{r},t) = \underbrace{\phi_{ni}(\underline{r},t)}_{Incident} \quad \underbrace{-\int_V d^3r_0 \int_{-\infty}^{\infty} dt_0 g(\underline{r},\underline{r}_0;t-t_0)\left\{\frac{\sigma_l(\underline{r}_0)}{c_0^2(\underline{r}_0)}\frac{\partial^2\phi(\underline{r}_0,t_0)}{\partial t_0^2} + \nabla\left(\gamma(\underline{r}_0)\nabla\phi(\underline{r}_0,t_0)\right)\right\}}_{linear\ scattering}$$
$$\underbrace{-\ 2\int_V d^3r_0 \int_{-\infty}^{\infty} dt_0 g(\underline{r},\underline{r}_0;t-t_0)\frac{\sigma_n(p;\underline{r}_0)}{c_0^2(\underline{r}_0)}\frac{\partial^2\phi_{ni}(\underline{r}_0,t_0)}{\partial t_0^2}}_{nonlinear\ scattering} \tag{1.13}$$

where $\phi_{ni}(\underline{r},t)$ is the surface integral in the theorem and represents the incident nonlinearly propagating wave. This wave can be simulated using the parabolic approximation of (1.10) in

(36). $g(\underline{r},\underline{r}_0;t)$ is the Green's function for the linear propagation operator and satisfies the linear propagation equation with an impulse point source as

$$\nabla^2 g(\underline{r},\underline{r}_0;t) - \frac{1}{c_0^2(\underline{r})}\frac{\partial^2 g(\underline{r},\underline{r}_0;t)}{\partial t^2} - h_p \underset{t}{\otimes} \frac{1}{c_0^2(\underline{r})}\frac{\partial^2 g(\underline{r},\underline{r}_0;t)}{\partial t^2} = -\delta(\underline{r}-\underline{r}_0)\delta(t) \tag{1.14}$$

The linear scattering integral includes the full wave field $\phi(\underline{r},t)$ and hence also includes multiple scattering. The Born approximation where we set $\phi(\underline{r},t) \approx \phi_{ni}(\underline{r},t)$ in the linear scattering term then presents the 1st order scattering. In the nonlinear scattering term we have done the approximation $\phi(\underline{r},t) \approx \phi_{ni}(\underline{r},t)$, because the nonlinear scattering is so weak that multiple scattering can be neglected. For numerical simulation of multiple scattering in a heterogeneous nonlinear tissue, it is convenient to do time integration of the wave equation as in (1.38).

The nonlinear elasticity in soft tissue can be described by the 2nd order approximation in (1.8). The nonlinear propagation velocity and the nonlinear scattering coefficient are then

$$\begin{gathered} c(\underline{r};p) = \frac{1}{\sqrt{\rho_a \kappa_a (1 - 2\beta_n \kappa_a p)}} \approx c_0 (1 + \beta_n \kappa_a p) \qquad c_0(\underline{r}) = \frac{1}{\sqrt{\rho_a \kappa_a}} \\ \sigma_n(\underline{r};p) = \left\{ 2\beta_{na}(2+\sigma_l)\sigma_l + \beta_{nf}(1+\sigma_l)^2 \right\} \kappa_a p \approx 4\beta_{na}\sigma_l \kappa_a p \end{gathered} \tag{1.15}$$

where the parameters are functions of $\underline{r}$. The wave equation, Eq.(1.12) then takes the form

$$\begin{aligned} &\underbrace{\nabla^2 \phi(\underline{r},t) - \frac{1}{c_0^2(\underline{r})}\frac{\partial^2 \phi(\underline{r},t)}{\partial t^2} + \frac{2\beta_{na}\sigma_l \kappa_a p}{c_0^2(\underline{r})}\frac{\partial^2 \phi(\underline{r},t)}{\partial t^2}}_{Nonlinear\ propagation} - \underbrace{h_p \underset{t}{\otimes} \frac{1}{c_0^2}\frac{\partial^2 \phi(\underline{r},t)}{\partial t^2}}_{absorption} \\ &= \underbrace{\frac{\sigma_l(\underline{r})}{c_0^2(\underline{r})}\frac{\partial^2 \phi(\underline{r},t)}{\partial t^2} + \nabla\left(\gamma(\underline{r})\nabla\phi(\underline{r},t)\right)}_{Linear\ scattering\ source\ terms} - \underbrace{\frac{4\beta_{na}\sigma_l \kappa_a p}{c_0^2(\underline{r})}\frac{\partial^2 \phi(\underline{r},t)}{\partial t^2}}_{Nonlinear\ scattering\ source\ term} \end{aligned} \tag{1.16}$$

Maximal values of the spatial fluctuation of the parameters are $\sigma_l \sim \pm 0.3$ and $\beta_{nf} \sim \pm 1.5$ [11] which allows the approximations as given. We hence see that the nonlinear scattering term is proportional to the linear scattering term σ_l with $4\beta_{na}\kappa_a p$ as proportionality constant. Typical values are $\sigma_l \sim \pm 0.1$ and $\beta_{nf} \sim \pm 0.5$ and gives $\sigma_n \sim 2\kappa_a p$. This allows the approximations of (1.17a), where we for soft tissue get a nonlinear scattering term that is proportional to the linear scattering term σ_l with $4\beta_{na}\kappa_a p$ as proportionality constant. For a solid particle in soft tissue, for example a micro-calcification one gets $\sigma_l \sim -1$ and $\beta_{na} \sim 5$. This gives a nonlinear scattering coefficent between the soft tissue and the solid particle as in (1.17b)

$$\sigma_n \approx 4\beta_{na}\sigma_l\kappa_a p \sim 2\kappa_a p \qquad \text{Soft tissue internal} \qquad a)$$

$$\sigma_n = 2\beta_{na}\kappa_a p \sim 10\kappa_a p \qquad \text{Solid particle in tissue} \qquad b) \tag{1.17}$$

$$\frac{\sigma_n \ part}{\sigma_n \ \text{int}} \sim 5 \sim 14dB \qquad c)$$

Hence, by adequate suppression of the linear scattering from the tissue, the nonlinear micro-calcification scattering to nonlinear tissue scattering has a contrast ratio ~ 14 dB. For $\kappa_a \approx 400 \cdot 10^{-12} \ Pa^{-1}$ and p = 2.5 MPa we get internally in the soft tissue

$$\frac{Nonl \ scat}{Lin \ scat} = \frac{(2+\sigma_l)\sigma_l}{\sigma_l - \gamma} 4\beta_{na}\kappa_a p \approx 4\beta_{na}\kappa_a p \sim 0.02 \sim -34dB \tag{1.18}$$

which suggests that the linearly scattered signal must be suppressed at least – 34dB for adequate nonlinear imaging of micro-calcifications in the tissue.

1.2 Wave equation in the pressure

We noet from Eq.(1.9) that $p(\underline{r},t) = \partial\phi(\underline{r},t)/\partial t$. Differentiating Eq.(1.16) with time, we note that

$$\frac{\partial}{\partial t}\left\{ p\frac{\partial^2\phi}{\partial t^2}\right\} = \frac{\partial}{\partial t}\left\{ p\frac{\partial p}{\partial t}\right\} = \frac{\partial p}{\partial t}\frac{\partial p}{\partial t} + p\frac{\partial^2 p}{\partial t^2} = \frac{1}{2}\frac{\partial^2 p^2}{\partial t^2} \tag{1.19}$$

which allows us to rearrange for an equation in the pressure only as

$$\begin{gathered}\underbrace{\nabla^2 p(\underline{r},t) - \frac{1}{c_0^2(\underline{r})}\frac{\partial^2 p(\underline{r},t)}{\partial t^2} + \frac{\beta_{na}\sigma_l\kappa_a}{c_0^2(\underline{r})}\frac{\partial^2 p^2(\underline{r},t)}{\partial t^2}}_{Nonlinear\ propagation} - \underbrace{h_p \underset{t}{\otimes} \frac{1}{c_0^2}\frac{\partial^2 p(\underline{r},t)}{\partial t^2}}_{absorption} \\ = \underbrace{\frac{\sigma_l(\underline{r})}{c_0^2(\underline{r})}\frac{\partial^2 p(\underline{r},t)}{\partial t^2} + \nabla\left(\gamma(\underline{r})\nabla p(\underline{r},t)\right)}_{Linear\ scattering\ source\ terms} - \underbrace{\frac{2\beta_{na}\sigma_l\kappa_a}{c_0^2(\underline{r})}\frac{\partial^2 p^2(\underline{r},t)}{\partial t^2}}_{\substack{Nonlinear\ scattering \\ source\ term}}\end{gathered} \tag{1.20}$$

1.3 Nonlinear interaction between composite band transmitted pulses

This is best studied with the simplified nonlinear elasticity of Eq.(1.15) as the nonlinear terms in Eqs.(1.19,20) are here a direct product form. Assume for example that we transmit a low and a high frequency pulse as

$$\widehat{\varphi}(\underline{r},t)=\widehat{\varphi}_L(\underline{r},t)+\widehat{\varphi}_H(\underline{r},t)$$

$$\widehat{p}(\underline{r},t)=\frac{\partial\widehat{\varphi}(\underline{r},t)}{\partial t}=\widehat{p}_L(\underline{r},t)+\widehat{p}_H(\underline{r},t) \tag{1.21}$$

$\widehat{\varphi}_L,\widehat{p}_L$ has frequencies around a low center frequency ω_L, while $\widehat{\varphi}_H,\widehat{p}_H$. The nonlinear source term then takes the form

$$\frac{\beta_n\kappa\widehat{p}}{c_0^2}\frac{\partial^2\widehat{\varphi}}{\partial t^2}=\frac{\beta_n\kappa}{c_0^2}\left(\widehat{p}_L+\widehat{p}_H\right)\frac{\partial^2\left(\widehat{\varphi}_L+\widehat{\varphi}_H\right)}{\partial t^2}$$

$$=\frac{\beta_n\kappa}{c_0^2}\left\{\widehat{p}_L\frac{\partial^2\widehat{\varphi}_L}{\partial t^2}+\widehat{p}_H\frac{\partial^2\widehat{\varphi}_L}{\partial t^2}+\widehat{p}_L\frac{\partial^2\widehat{\varphi}_H}{\partial t^2}+\widehat{p}_H\frac{\partial^2\widehat{\varphi}_H}{\partial t^2}\right\}$$

$$=\frac{\beta_n\kappa}{c_0^2}\left\{\frac{\partial\widehat{\varphi}_L}{\partial t}\frac{\partial^2\widehat{\varphi}_L}{\partial t^2}+\frac{\partial\widehat{\varphi}_H}{\partial t}\frac{\partial^2\widehat{\varphi}_L}{\partial t^2}+\frac{\partial\widehat{\varphi}_L}{\partial t}\frac{\partial^2\widehat{\varphi}_H}{\partial t^2}+\frac{\partial\widehat{\varphi}_H}{\partial t}\frac{\partial^2\widehat{\varphi}_H}{\partial t^2}\right\} \tag{1.22}$$

For further analysis it is convenient to use the complex representation of these signals

$$\widehat{\varphi}(\underline{r},t)=\mathrm{Re}\left\{\hat{\varphi}(\underline{r},t)\right\}=\mathrm{Re}\left\{\tilde{\varphi}(\underline{r},t)e^{i\omega_c t}\right\}=\left|\tilde{\varphi}(\underline{r},t)\right|\cos\left(\omega_c t+\theta_{\tilde{\varphi}}(\underline{r},t)\right)$$

$$\widehat{p}(\underline{r},t)=\mathrm{Re}\left\{\hat{p}(\underline{r},t)\right\}=\mathrm{Re}\left\{\tilde{p}(\underline{r},t)e^{i\omega_c t}\right\}=\left|\tilde{p}(\underline{r},t)\right|\cos\left(\omega_c t+\theta_{\tilde{p}}(\underline{r},t)\right)$$

$$\begin{aligned}\hat{\varphi}(\underline{r},t)&=\tilde{\varphi}(\underline{r},t)e^{i\omega_c t} &\qquad \hat{\varphi}(\underline{r},t)&=\left|\tilde{\varphi}(\underline{r},t)\right|e^{i\theta_{\tilde{\varphi}}(\underline{r},t)}\\ \hat{p}(\underline{r},t)&=\tilde{p}(\underline{r},t)e^{i\omega_c t} &\qquad \hat{p}(\underline{r},t)&=\left|\tilde{p}(\underline{r},t)\right|e^{i\theta_{\tilde{p}}(\underline{r},t)}\end{aligned} \tag{1.23}$$

where ω_c is the center frequency of the band. $\hat{\varphi}(\underline{r},\tau),\hat{p}(\underline{r},\tau)$ are termed the analytic reoresentation of the real pulses, and $\tilde{\varphi}(\underline{r},\tau),\tilde{p}(\underline{r},\tau)$ are the complex envelopes of the pulses. The product of two real functions are represented by the analytic signals as

$$\mathrm{Re}\left\{\hat{\varphi}_L\right\}\mathrm{Re}\left\{\hat{\varphi}_H\right\}=\frac{1}{2}\left(\hat{\varphi}_L+\hat{\varphi}_L^*\right)\frac{1}{2}\left(\hat{\varphi}_H+\hat{\varphi}_H^*\right)=\frac{1}{4}\left(\hat{\varphi}_L\hat{\varphi}_H+\hat{\varphi}_L^*\hat{\varphi}_H^*\right)+\frac{1}{4}\left(\hat{\varphi}_L^*\hat{\varphi}_H+\hat{\varphi}_L\hat{\varphi}_H^*\right)$$

$$=\frac{1}{2}\mathrm{Re}\left\{\hat{\varphi}_L\hat{\varphi}_H\right\}+\frac{1}{2}\mathrm{Re}\left\{\hat{\varphi}_L^*\hat{\varphi}_H\right\}=\frac{1}{2}\mathrm{Re}\left\{\tilde{\varphi}_L\tilde{\varphi}_He^{i(\omega_H+\omega_L)t}\right\}+\frac{1}{2}\mathrm{Re}\left\{\tilde{\varphi}_L^*\tilde{\varphi}_He^{i(\omega_H-\omega_L)t}\right\} \tag{1.24}$$

We note that $\hat{\varphi}_L\hat{\varphi}_H$ produces the sum of the frequency $\omega_H+\omega_L$ while $\hat{\varphi}_L^*\hat{\varphi}_H$ produces the difference frequency $\omega_H-\omega_L$.

Returning to Eq.(1.22) we see that the nonlinear terms $\left(\partial\widehat{\varphi}_L/\partial t\right)\left(\partial^2\widehat{\varphi}_L/\partial t^2\right)$ in Eq.(1.16) produces self distortion of the low frequency pulse that introduces harmonics of the low frequency ω_L. The amplitude of this 1st term in Eq.(1.16, 22) is due to the differentiation proportional to ω_L^3. The last term $\left(\partial\widehat{\varphi}_H/\partial t\right)\left(\partial^2\widehat{\varphi}_H/\partial t^2\right)$ produces self distortion of the high frequency pulse that introduces harmonics of the high frequency ω_H. The amplitude of the last term is proportional to ω_H^3.

The two middle nonlinear source terms produce interaction between the low and high frequency pulses, and introduces sum and difference frequencies between the low and high frequency components. The 2nd term introduces 2nd order differentaion of $\widehat{\varphi}_L$ and 1st order differentiation of $\widehat{\varphi}_H$ which has an amplitude $\sim\omega_H\omega_L^2$, while the 3rd term includes 2nd order differentiation of $\widehat{\varphi}_H$ which has amplitude ~ $\omega_L\omega_H^2$. For a ratio of the LF to HF of $\omega_L:\omega_H$ □ $1:10$, the terms in Eq.(1.22) have the following amplitude ratio to each other 1:10:100:1000. We hence can neglect the 2nd term for the interaction between the LA and HF.

For the pressure equation, Eq.(1.20) we note that

$$\frac{\partial^2\left(\widehat{p}_L+\widehat{p}_H\right)^2}{\partial t^2}=\frac{\partial^2\widehat{p}_L^2}{\partial t^2}+2\frac{\partial^2\widehat{p}_L\widehat{p}_H}{\partial t^2}+\frac{\partial^2\widehat{p}_H^2}{\partial t^2} \tag{1.25}$$

where the 1st term represents the nonlinear self distortion of the LF pulse, the last term represents the nonlinear self distortion of the HF pulse, and the middle term represents the nonlinear interaction between the two pulses. We note from Eq.(1.24) that the

Through the differentiation, the 1st term is □ $2\omega_L^2$, the 2nd term□ $\left(\omega_H-\omega_L\right)^2$ and $\left(\omega_H+\omega_L\right)^2$, and the last term is □ $2\omega_H^2$. With a ratio 1:10 between the LF and HF, the ratios of the amplitude of terms are 1:100:100.

2. Numerical simulation of forward wave propagation

2.1 Retarded time equation for momentum potential

Neglecting the scattering in Eq.(1.16) we get

$$\nabla^2\varphi\left(\underline{r},t\right)-\frac{1}{c_0^2}\frac{\partial^2\varphi\left(\underline{r},t\right)}{\partial t^2}+\frac{2\beta_n(\underline{r})\kappa(\underline{r})p\left(\underline{r},t\right)}{c_0^2}\frac{\partial^2\varphi\left(\underline{r},t\right)}{\partial t^2}-\frac{1}{c^2\left(p\left(\underline{r},t\right)\right)}h\underset{t}{\otimes}\frac{\partial^2\varphi\left(\underline{r},t\right)}{\partial t^2}=0$$

Inserts retarded time τ

$$\tau=t-\frac{z}{c_0}\qquad t=\tau+\frac{z}{c_0}\qquad \varphi\left(z,\underline{r}_\perp,t\right)\to\widehat{\varphi}\left(z,\underline{r}_\perp,\tau\right)\qquad p\left(z,\underline{r}_\perp,t\right)\to\widehat{p}\left(z,\underline{r}_\perp,\tau\right)$$

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \widehat{\varphi}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial \widehat{\varphi}}{\partial \tau} \qquad \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \widehat{\varphi}}{\partial \tau^2}$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \widehat{\varphi}}{\partial z} + \frac{\partial \widehat{\varphi}}{\partial \tau}\frac{\partial \tau}{\partial z} \qquad \frac{\partial^2 \varphi}{\partial z^2} = \frac{\partial^2 \widehat{\varphi}}{\partial z^2} + 2\frac{\partial^2 \widehat{\varphi}}{\partial z \partial \tau}\frac{\partial \tau}{\partial z} + \frac{\partial^2 \widehat{\varphi}}{\partial \tau^2}\left(\frac{\partial \tau}{\partial z}\right)^2$$

$$\frac{\partial^2 \varphi}{\partial z^2} = \frac{\partial^2 \widehat{\varphi}}{\partial z^2} - \frac{2}{c_0}\frac{\partial^2 \widehat{\varphi}}{\partial z \partial \tau} + \frac{1}{c_0^2}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2}$$

$$\frac{\partial^2 \widehat{\varphi}}{\partial z^2} - \frac{2}{c_0}\frac{\partial^2 \widehat{\varphi}}{\partial z \partial \tau} + \frac{1}{c_0^2}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} + \nabla_\perp^2 \widehat{\varphi} - \frac{1}{c_0^2}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} + \frac{2\beta_n(\underline{r})\kappa(\underline{r})\widehat{p}\left(z,\underline{r}_\perp,\tau\right)}{c_0^2}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} - \frac{1}{c_0^2}h \underset{t}{\otimes} \frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} = 0$$

$$\frac{\partial^2 \widehat{\varphi}}{\partial z^2} - \frac{2}{c_0}\frac{\partial^2 \widehat{\varphi}}{\partial z \partial \tau} + \nabla_\perp^2 \widehat{\varphi} + \frac{2\beta_n(\underline{r})\kappa(\underline{r})\widehat{p}\left(z,\underline{r}_\perp,\tau\right)}{c_0^2}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} - \frac{1}{c_0^2}h \underset{t}{\otimes} \frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} = 0$$

Approximates $\partial^2 \widehat{\varphi} / \partial z^2 \approx 0$ which gives the paraxial approximation of the nonlinear propagation equation

$$\frac{\partial^2 \widehat{\varphi}}{\partial z \partial \tau} = \frac{c_0}{2}\nabla_\perp^2 \widehat{\varphi} - h \underset{t}{\otimes} \frac{1}{c_0}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2} + \beta_n(\underline{r})\kappa(\underline{r})\widehat{p}\left(z,\underline{r}_\perp,\tau\right)\frac{1}{c_0}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2}$$

The LF pressure at $\left(z,\underline{r}_\perp,\tau\right)$ of the HF pulse is

$$\widehat{p}_L\left(z,\underline{r}_\perp,\tau\right) = \widehat{p}_L\left(z,\underline{r}_\perp,0\right)\cos\omega_L\left(\tau + \tau_L(z,\underline{r}_\perp)\right)$$

where $\tau_L(z,\underline{r}_\perp)$ is the location of the center of the HF pulse witin the LF pulse. We define

$$\alpha\left(z,\underline{r}_\perp\right) = \beta_n\left(z,\underline{r}_\perp\right)\kappa\left(z,\underline{r}_\perp\right)\widehat{p}_L\left(z,\underline{r}_\perp,0\right)$$

and from Eq.(XX) we can then write an Equation that takes into account the nonlinear propoagation delay and pulse form distortion that is inflicted upon the HF pulse by the LF pulse plus the self distortion of the HF pulse (two last terms in Eq.(XX)).

$$\begin{aligned}\frac{\partial^2 \widehat{\varphi}\left(z,\underline{r}_\perp,\tau\right)}{\partial z \partial \tau} &= \frac{c_0}{2}\nabla_\perp^2 \widehat{\varphi}\left(z,\underline{r}_\perp,\tau\right) - h \underset{t}{\otimes} \frac{1}{c_0}\frac{\partial^2 \widehat{\varphi}\left(z,\underline{r}_\perp,\tau\right)}{\partial \tau^2} \\ &+\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\left(\tau + \tau_L(z,\underline{r}_\perp)\right)\frac{1}{c_0}\frac{\partial^2 \widehat{\varphi}\left(z,\underline{r}_\perp,\tau\right)}{\partial \tau^2} + \frac{\alpha\left(z,\underline{r}_\perp\right)\widehat{p}\left(z,\underline{r}_\perp,\tau\right)}{\widehat{p}_L\left(z,\underline{r}_\perp,0\right)}\frac{1}{c_0}\frac{\partial^2 \widehat{\varphi}}{\partial \tau^2}\end{aligned}$$

where $\widehat{p}\left(z,\underline{r}_\perp,\tau\right) = \partial \widehat{\varphi}\left(z,\underline{r}_\perp,\tau\right)/\partial \tau$ is the HF pulse. The 3rd term on the right side produces nonlinear delay and pulse form distortion. The last term is responsible for the nonlinear self distortion of the HF pulse.

2.2 Retarded time equation for the pressure pulse

Neglecting the scattering terms in Eq.(1.20), the retarded time equation takes the form

$$\frac{\partial^2 \hat{p}}{\partial z \partial \tau} = \frac{c_0}{2} \nabla_\perp^2 \hat{p} - h \underset{t}{\otimes} \frac{1}{c_0} \frac{\partial^2 \hat{p}}{\partial \tau^2} + \frac{\beta_n \kappa}{2c_0} \frac{\partial^2 \hat{p}^2}{\partial \tau^2}$$

With the composite LF and HF pulses the nonlinear source term takes the form

$$\frac{\partial^2 \left(\hat{p}_L + \hat{p}_H \right)^2}{\partial \tau^2} = \frac{\partial^2 \hat{p}_L^2}{\partial \tau^2} + 2 \frac{\partial^2 \hat{p}_L \hat{p}_H}{\partial \tau^2} + \frac{\partial^2 \hat{p}_H^2}{\partial \tau^2}$$

with the arguments from above the 1st term represents the self distortion of the LF pulse, the 2nd represents the interaction between the HF and LF pulses, while the last term represents the self distortion of the HF pulse.

Comment: The term for interaction between the HF and LF pulses can be further developed to

$$\frac{\partial^2 \hat{p}_L \hat{p}_H}{\partial \tau^2} = \hat{p}_H \frac{\partial^2 \hat{p}_L}{\partial \tau^2} + 2 \frac{\partial \hat{p}_L}{\partial \tau} \frac{\partial \hat{p}_H}{\partial \tau} + \hat{p}_L \frac{\partial^2 \hat{p}_H}{\partial \tau^2}$$

Through the differentiation the 1st term is □ ω_L^2, the 2nd term is □ $\omega_L \omega_H$, and the last term is □ ω_H^2. With a ratio 1:10 between the LF and HF, the ratios of the terms are 1:20:100. The 2nd term is hemce 1:5 of the last term, which is on the border of neglecting. However, we shall see in the next paragraph that there is now computational gain in doing such an approximation.

Equation for HF pulse: Using only the last two terms of this expression, the pressure equation for the HF pulse takes the form

$$\frac{\partial^2 \hat{p}}{\partial z \partial \tau} = \frac{c_0}{2} \nabla_\perp^2 \hat{p} - h \underset{t}{\otimes} \frac{1}{c_0} \frac{\partial^2 \hat{p}}{\partial \tau^2} + \frac{\beta_n \kappa}{c_0} \frac{\partial^2 \hat{p}_L \hat{p}}{\partial \tau^2} + \frac{\beta_n \kappa}{2c_0} \frac{\partial^2 \hat{p}^2}{\partial \tau^2}$$

where the variable $\hat{p}$ that represents the HF pulse. The last term produces the nonlinear self-distortion of the HF pulse, while the 2nd last term produces the modification (nonlinear propagation delay and pulse form distortion) of the HF pulse by the LF pulse.

Equation for LF pulse: Using the 1st term of Eq.(XX) we get the following equation for the LF pulse with nonlinear self distortion

$$\frac{\partial^2 \hat{p}_L}{\partial z \partial \tau} = \frac{c_0}{2} \nabla_\perp^2 \hat{p}_L - h \underset{t}{\otimes} \frac{1}{c_0} \frac{\partial^2 \hat{p}_L}{\partial \tau^2} + \frac{\beta_n \kappa}{2c_0} \frac{\partial^2 \hat{p}_L^2}{\partial \tau^2}$$

The nonlinear self distortion of the LF pulse produces a triangular distortion of the pulse oscillations which through the influence of diffraction produces a peaking of the positive swing of the LF pulse and a flattening of the negative LF swing [XX]. This distortion hence modifies the LF amplitude that is observed by the HF pulse for positive and negative LF pulses. However, the LF pulse propagates only ~ 10 – 20 LF wavelengths for $\omega_L : \omega_H \sim 1:10$. This limits the nonlinear self distortion of the LF pulse, where one can approximate the propagation to be linear.

2.2 Temporal FT of the HF equation

We start with Eq.(XX) and use an interacting low pressure pulse as in Eq.(XX) as

$$\beta_n\left(z,\underline{r}_\perp\right)\kappa\left(z,\underline{r}_\perp\right)\hat{p}_L\left(z,\underline{r}_\perp,\tau\right)=\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\left(\tau+\tau_L(z,\underline{r}_\perp)\right)$$

$$\alpha\left(z,\underline{r}_\perp\right)=\beta_n\left(z,\underline{r}_\perp\right)\kappa\left(z,\underline{r}_\perp\right)\hat{p}_L\left(z,\underline{r}_\perp,0\right)$$

Temporal Fourier transform of the LF HF interaction term in Eq.(XX) is then

$$F\left\{\frac{\beta_n\kappa}{c_0}\frac{\partial^2\hat{p}_L\hat{p}_H}{\partial\tau^2}\right\}=\frac{\alpha\left(z,\underline{r}_\perp\right)}{c_0}\int d\tau\ e^{-i\omega\tau}\frac{\partial^2}{\partial\tau^2}\left[\cos\omega_L\left(\tau+\tau_L(z,\underline{r}_\perp)\right)\hat{p}\left(z,\underline{r}_\perp,\tau\right)\right]$$

$$=\frac{\alpha}{2c_0}\int d\tau\ e^{-i\omega\tau}\frac{\partial^2}{\partial\tau^2}\left[e^{i\omega_L\left(\tau+\tau_L(z,\underline{r}_\perp)\right)}\hat{p}\left(z,\underline{r}_\perp,\tau\right)\right]+\frac{\alpha}{2c_0}\int d\tau\ e^{-i\omega\tau}\frac{\partial^2}{\partial\tau^2}\left[e^{-i\omega_L\left(\tau+\tau_L(z,\underline{r}_\perp)\right)}\hat{p}\left(z,\underline{r}_\perp,\tau\right)\right]$$

$$=-\frac{\alpha\omega^2}{2c_0}\left\{e^{i\omega_L\tau_L(z,\underline{r}_\perp)}\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)+e^{-i\omega_L\tau_L(z,\underline{r}_\perp)}\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)\right\}$$

Full temporal Fourier transform of Eq.(XX) then takes the form

$$i\omega\frac{d\hat{p}\left(z,\underline{r}_\perp,\omega\right)}{dz}=\frac{c_0}{2}\nabla_\perp^2\hat{p}\left(z,\underline{r}_\perp,\omega\right)+\frac{\omega^2}{2c_0}H\left(z,\underline{r}_\perp,\omega\right)\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

$$-\frac{\alpha\left(z,\underline{r}_\perp\right)\omega^2}{2c_0}\left\{e^{i\omega_L\tau_L(z,\underline{r}_\perp)}\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)+e^{-i\omega_L\tau_L(z,\underline{r}_\perp)}\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)\right\}$$

$$-\frac{\alpha\left(z,\underline{r}_\perp\right)\omega^2}{2c_0\hat{p}_L\left(z,\underline{r}_\perp,0\right)}\hat{p}\left(z,\underline{r}_\perp,\omega\right)\underset{\omega}{\otimes}\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

where the last term represents convolution in the frequency domain

$$\hat{p}\left(z,\underline{r}_\perp,\omega\right)\underset{\omega}{\otimes}\hat{p}\left(z,\underline{r}_\perp,\omega\right)=\frac{1}{2\pi}\int dw\ \hat{p}\left(\omega-w\right)\hat{p}\left(w\right)$$

Eq.(XX) is further modified to

$$\frac{d\hat{p}\left(z,\underline{r}_\perp,\omega\right)}{dz} = -i\frac{c_0}{2\omega}\nabla_\perp^2\hat{p}\left(z,\underline{r}_\perp,\omega\right) - i\frac{\omega}{2c_0}H\left(z,\underline{r}_\perp,\omega\right)\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

$$+\frac{i}{2}\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\left\{e^{i\omega_L\tau_L(z,\underline{r}_\perp)}\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right) + e^{-i\omega_L\tau_L(z,\underline{r}_\perp)}\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)\right\}$$

$$+\frac{i}{2}\frac{\omega}{c_0}\frac{\alpha\left(z,\underline{r}_\perp\right)}{\hat{p}_L\left(z,\underline{r}_\perp,0\right)}\hat{p}\left(z,\underline{r}_\perp,\omega\right)\underset{\omega}{\otimes}\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

which also can be written in the form

$$\frac{d\hat{p}\left(z,\underline{r}_\perp,\omega\right)}{dz} = -i\frac{c_0}{2\omega}\nabla_\perp^2\hat{p}\left(z,\underline{r}_\perp,\omega\right) - i\frac{\omega}{2c_0}H\left(z,\underline{r}_\perp,\omega\right)\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

$$+i\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\tau_L(z,\underline{r}_\perp)\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

$$+\frac{i}{2}\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\tau_L(z,\underline{r}_\perp)\left\{\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)+\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)-2\hat{p}\left(z,\underline{r}_\perp,\omega\right)\right\}$$

$$-\frac{1}{2}\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\sin\omega_L\tau_L(z,\underline{r}_\perp)\left\{\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)-\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)\right\}$$

$$+\frac{i}{2}\frac{\omega}{c_0}\frac{\alpha\left(z,\underline{r}_\perp\right)}{\hat{p}_L\left(z,\underline{r}_\perp,0\right)}\hat{p}\left(z,\underline{r}_\perp,\omega\right)\underset{\omega}{\otimes}\hat{p}\left(z,\underline{r}_\perp,\omega\right)$$

The first term on the right side represents diffraction, the 2^{nd} term absorption, the 3^{rd} term the nonlinear propagation delay of the center of the HF pulse due to the LF pulse, the 4^{th} term pulse form distortion of the HF pulse due to the curvature of the LF pulse which produces a nonlinear phase distortion, the 5^{th} term pulse form distortion of the HF pulse due to the gradient of the LF pulse around the center of the HF pulse which produces a spectral amplitude distortion, and the 6^{th} term represents the self distortion of the HF pulse that generates harmonics and sub harmonics of the HF frequency. In the receiver one would typically band pass filter the received signal around the fundamental frequency. The last term hence represents a nonlinear absorption attenuation of the fundamental HF band by pumping energy into the harmonic and sub-harmonic components of the HF band.

The frequency offsets of the pressure spectra can be approximated by Taylor series

$$\frac{\hat{p}\left(s,\underline{r}_{\perp},\omega-\omega_L\right)+\hat{p}\left(s,\underline{r}_{\perp},\omega+\omega_L\right)}{2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}-1$$

$$\approx\frac{1}{2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}\frac{\partial\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}{\partial\omega}\left(-\omega_L+\omega_L\right)+\frac{1}{4\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}\frac{\partial^2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}{\partial\omega^2}\left(\omega_L^2+\omega_L^2\right)$$

$$=\frac{\omega_L^2}{2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}\frac{\partial^2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}{\partial\omega^2}$$

and

$$\frac{\hat{p}\left(s,\underline{r}_{\perp},\omega-\omega_L\right)-\hat{p}\left(s,\underline{r}_{\perp},\omega+\omega_L\right)}{2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}\approx\frac{1}{2\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}\frac{\partial\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}{\partial\omega}\left(-\omega_L-\omega_L\right)$$

$$=\frac{\omega_L}{\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}\frac{\partial\hat{p}\left(s,\underline{r}_{\perp},\omega\right)}{\partial\omega}$$

3. Numerical integration of the equation

3.1 Operator splitting and Burgers equation

The last term in Eqs.(XX,XX) represents self distortion of the HF pulse and introduces harmonic components of the HF pulse fundamental band which in its turn produces a nonlinear attenuation of the fundamental band of the HF pulse.

In most situations one is mainly interested in the fundamental HF band, and in case the nonlinear attenuation can be neglected, the last term in Eqs.(XX,XX) can be neglected, which greatly simplifies the solution. The nonlinear self distortion can be included in numerical solutions of Eq.(XX) through a method called operator splitting. We first provide a solution for the nonlinear HF self distortion

$$\frac{\partial^2\overline{p}}{\partial z\partial\tau}=\frac{\beta_n\kappa}{2c_0}\frac{\partial^2\overline{p}^2}{\partial\tau^2}\qquad\Rightarrow\qquad\frac{\partial\overline{p}}{\partial z}-\frac{\beta_n\kappa\overline{p}}{c_0}\frac{\partial\overline{p}}{\partial\tau}=0\qquad\textit{Burgers Equation}$$

and this solution is introduced into the right side as

$$\frac{\partial^2\hat{p}}{\partial z\partial\tau}=\frac{c_0}{2}\nabla_{\perp}^2\overline{p}-h\underset{t}{\otimes}\frac{1}{c_0}\frac{\partial^2\overline{p}}{\partial\tau^2}+\frac{\beta_n\kappa}{c_0}\frac{\partial^2\hat{p}_L\overline{p}}{\partial\tau^2}$$

The Burgers Equation is solved through the method of characteristics

$$\overline{p}\left(z,\tau\right)=f\left(\tau+\beta_n\kappa\overline{p}\,z/c_0\right)=\hat{p}\left(z_{i-1},\underline{r}_{\perp},\tau+\beta_n\kappa\overline{p}\,z/c_0\right)$$

The 2nd equation can be integrated directly

$$d\widehat{p}\left(z,\underline{r}_\perp,\omega\right) = -i\left\{\frac{c_0}{2\omega}\nabla_\perp^2\widehat{p}\left(z,\underline{r}_\perp,\omega\right) + \frac{\omega}{2c_0}H\left(z,\underline{r}_\perp,\omega\right)\widehat{p}\left(z,\underline{r}_\perp,\omega\right)\right\}dz$$

$$+i\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\tau_L(z,\underline{r}_\perp)\widehat{p}\left(z,\underline{r}_\perp,\omega\right)dz$$

$$+\frac{i}{2}\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\tau_L(z,\underline{r}_\perp)\left\{\widehat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)+\widehat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)-2\widehat{p}\left(z,\underline{r}_\perp,\omega\right)\right\}dz$$

$$-\frac{1}{2}\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\sin\omega_L\tau_L(z,\underline{r}_\perp)\left\{\widehat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)-\widehat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)\right\}dz$$

which can be turned into an integral equation as

$$\widehat{p}\left(z,\underline{r}_\perp,\omega\right) = \widehat{p}\left(0,\underline{r}_\perp,\omega\right) - i\int_0^z\left\{\frac{c_0}{2\omega}\nabla_\perp^2\widehat{p}\left(s,\underline{r}_\perp,\omega\right) + \frac{\omega}{2c_0}H\left(s,\underline{r}_\perp,\omega\right)\widehat{p}\left(s,\underline{r}_\perp,\omega\right)\right\}ds$$
$$+i\frac{\omega}{c_0}\int_0^z ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)\widehat{p}\left(s,\underline{r}_\perp,\omega\right)$$
$$+\frac{i}{2}\frac{\omega}{c_0}\int_0^z ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)\left\{\widehat{p}\left(s,\underline{r}_\perp,\omega-\omega_L\right)+\widehat{p}\left(s,\underline{r}_\perp,\omega+\omega_L\right)-2\widehat{p}\left(s,\underline{r}_\perp,\omega\right)\right\}$$
$$-\frac{1}{2}\frac{\omega}{c_0}\int_0^z ds\ \alpha\left(s,\underline{r}_\perp\right)\sin\omega_L\tau_L(s,\underline{r}_\perp)\left\{\widehat{p}\left(s,\underline{r}_\perp,\omega-\omega_L\right)-\widehat{p}\left(s,\underline{r}_\perp,\omega+\omega_L\right)\right\}ds$$

When the integration interval (0,z) is large, the equation can be solved through iteration, as

$$\widehat{p}_{n+1}\left(z,\underline{r}_\perp,\omega\right) = \widehat{p}\left(0,\underline{r}_\perp,\omega\right) - i\int_0^z\left\{\frac{c_0}{2\omega}\nabla_\perp^2\widehat{p}_n\left(s,\underline{r}_\perp,\omega\right) + \frac{\omega}{2c_0}H\left(s,\underline{r}_\perp,\omega\right)\widehat{p}_n\left(s,\underline{r}_\perp,\omega\right)\right\}ds$$
$$+i\frac{\omega}{c_0}\int_0^z ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)\widehat{p}_n\left(s,\underline{r}_\perp,\omega\right)$$
$$+\frac{i}{2}\frac{\omega}{c_0}\int_0^z ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)\left\{\widehat{p}_n\left(s,\underline{r}_\perp,\omega-\omega_L\right)+\widehat{p}_n\left(s,\underline{r}_\perp,\omega+\omega_L\right)-2\widehat{p}_n\left(s,\underline{r}_\perp,\omega\right)\right\}$$
$$-\frac{1}{2}\frac{\omega}{c_0}\int_0^z ds\ \alpha\left(s,\underline{r}_\perp\right)\sin\omega_L\tau_L(s,\underline{r}_\perp)\left\{\widehat{p}_n\left(s,\underline{r}_\perp,\omega-\omega_L\right)-\widehat{p}_n\left(s,\underline{r}_\perp,\omega+\omega_L\right)\right\}ds$$

or we can use a set of small integration step using the Euler approximation to the derivative to get

$$\hat{p}\left(z_{n+1},\underline{r}_\perp,\omega\right)=\hat{p}\left(z_n,\underline{r}_\perp,\omega\right)-i\left\{\frac{c_0}{2\omega}\nabla_\perp^2\hat{p}\left(z_n,\underline{r}_\perp,\omega\right)+\frac{\omega}{2c_0}H\left(z_n,\underline{r}_\perp,\omega\right)\hat{p}\left(z_n,\underline{r}_\perp,\omega\right)\right\}\Delta z$$

$$+i\frac{\omega}{c_0}\alpha\left(z_n,\underline{r}_\perp\right)\cos\omega_L\tau_L\left(z_n,\underline{r}_\perp\right)\hat{p}\left(z_n,\underline{r}_\perp,\omega\right)\Delta z$$

$$+\frac{i}{2}\frac{\omega}{c_0}\alpha\left(z_n,\underline{r}_\perp\right)\cos\omega_L\tau_L\left(z_n,\underline{r}_\perp\right)\left\{\hat{p}\left(z_n,\underline{r}_\perp,\omega-\omega_L\right)+\hat{p}\left(z_n,\underline{r}_\perp,\omega+\omega_L\right)-2\hat{p}\left(z_n,\underline{r}_\perp,\omega\right)\right\}\Delta z$$

$$-\frac{1}{2}\frac{\omega}{c_0}\alpha\left(z_n,\underline{r}_\perp\right)\sin\omega_L\tau_L\left(z_n,\underline{r}_\perp\right)\left\{\hat{p}\left(z_n,\underline{r}_\perp,\omega-\omega_L\right)-\hat{p}\left(z_n,\underline{r}_\perp,\omega+\omega_L\right)\right\}\Delta z$$

A more efficient integration can be found through the following procedure

$$\frac{d\hat{p}\left(z,\underline{r}_\perp,\omega\right)}{\hat{p}\left(z,\underline{r}_\perp,\omega\right)}=\left\{-i\frac{c_0}{2\omega}\frac{\nabla_\perp^2\hat{p}\left(z,\underline{r}_\perp,\omega\right)}{\hat{p}\left(z,\underline{r}_\perp,\omega\right)}-i\frac{\omega}{2c_0}H\left(z,\underline{r}_\perp,\omega\right)+i\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\tau_L(z,\underline{r}_\perp)\right\}dz$$

$$+i\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\cos\omega_L\tau_L(z,\underline{r}_\perp)\left\{\frac{\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)+\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)}{2\hat{p}\left(z,\underline{r}_\perp,\omega\right)}-1\right\}dz$$

$$-i\frac{\omega}{c_0}\alpha\left(z,\underline{r}_\perp\right)\sin\omega_L\tau_L(z,\underline{r}_\perp)\frac{\hat{p}\left(z,\underline{r}_\perp,\omega-\omega_L\right)-\hat{p}\left(z,\underline{r}_\perp,\omega+\omega_L\right)}{2\hat{p}\left(z,\underline{r}_\perp,\omega\right)}dz$$

which is integrated to

$$\hat{p}\left(z_i,\underline{r}_\perp,\omega\right)=\hat{p}\left(z_{i-1},\underline{r}_\perp,\omega\right)\exp\left\{-i\frac{c_0}{2\omega}\int_{z_{i-1}}^{z_i}ds\frac{\nabla_\perp^2\hat{p}\left(s,\underline{r}_\perp,\omega\right)}{\hat{p}\left(s,\underline{r}_\perp,\omega\right)}-i\frac{\omega}{2c_0}\int_{z_{i-1}}^{z_i}dsH\left(s,\underline{r}_\perp,\omega\right)-i\omega\delta\tau_n(z_i,\underline{r}_\perp)\right\}$$

$$\times\exp\left\{i\frac{\omega}{c_0}\int_{z_{i-1}}^{z_i}ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L\left(s,\underline{r}_\perp\right)\left(\frac{\hat{p}\left(s,\underline{r}_\perp,\omega-\omega_L\right)+\hat{p}\left(s,\underline{r}_\perp,\omega+\omega_L\right)}{2\hat{p}\left(s,\underline{r}_\perp,\omega\right)}-1\right)\right\}$$

$$\times\exp\left\{-\frac{\omega}{c_0}\int_{z_{i-1}}^{z_i}ds\ \alpha\left(s,\underline{r}_\perp\right)\sin\omega_L\tau_L\left(s,\underline{r}_\perp\right)\frac{\hat{p}\left(s,\underline{r}_\perp,\omega-\omega_L\right)-\hat{p}\left(s,\underline{r}_\perp,\omega+\omega_L\right)}{2\hat{p}\left(s,\underline{r}_\perp,\omega\right)}\right\}$$

where the term with the nonlinear propagation delay is defined through

$$\delta\tau_n(z_i,\underline{r}_\perp)=-\frac{1}{c_0}\int_{z_{i-1}}^{z_i}ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)\qquad\tau_n(z_i,\underline{r}_\perp)=-\frac{1}{c_0}\int_0^{z_i}ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)$$

We note that Eq.(XX) is an integral equation, as the pressure function $\hat{p}\left(s,\underline{r}_\perp,\omega\right)$ is found under the integrals on the right side. The convolution term also introduces a nonlinear double integral term, which complicates the situation. One can use an approximate solution method where for example we set $\widehat{p}\left(s,\underline{r}_\perp,\omega\right)=\widehat{p}\left(z_{i-1},\underline{r}_\perp,\omega\right)$ (Born type approximation). This limits the useful integration interval. The integration interval can be increased through iteration of the integral equation, where the first step uses $\widehat{p}_0\left(s,\underline{r}_\perp,\omega\right)=\widehat{p}\left(z_{i-1},\underline{r}_\perp,\omega\right)$ on the right side to produce $\widehat{p}_0\left(z_i,\underline{r}_\perp,\omega\right)$, and for further iteration steps we set $\widehat{p}_n\left(s,\underline{r}_\perp,\omega\right)=\widehat{p}_{n-1}\left(z_i,\underline{r}_\perp,\omega\right)$ for the right side integrals.

With the Taylor series approximation of the frequency offsets, we approximate

$$\widehat{p}\left(z_i,\underline{r}_\perp,\omega\right)=\widehat{p}\left(z_{i-1},\underline{r}_\perp,\omega\right)\exp\left\{-i\frac{c_0}{2\omega}\int_{z_{i-1}}^{z_i}ds\frac{\nabla_\perp^2\widehat{p}\left(s,\underline{r}_\perp,\omega\right)}{\widehat{p}\left(s,\underline{r}_\perp,\omega\right)}-i\frac{\omega}{2c_0}\int_{z_{i-1}}^{z_i}dsH\left(s,\underline{r}_\perp,\omega\right)-i\omega\delta\tau_n(z_i,\underline{r}_\perp)\right\}$$

$$\times\exp\left\{ik_\omega\int_{z_{i-1}}^{z_i}ds\ \alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L\left(s,\underline{r}_\perp\right)\frac{\omega_L^2}{2\widehat{p}\left(s,\underline{r}_\perp,\omega\right)}\frac{\partial^2\widehat{p}\left(s,\underline{r}_\perp,\omega\right)}{\partial\omega^2}\right\}$$

$$\times\exp\left\{-k_\omega\int_{z_{i-1}}^{z_i}ds\ \alpha\left(s,\underline{r}_\perp\right)\sin\omega_L\tau_L\left(s,\underline{r}_\perp\right)\frac{\omega_L}{\widehat{p}\left(s,\underline{r}_\perp,\omega\right)}\frac{\partial\widehat{p}\left(s,\underline{r}_\perp,\omega\right)}{\partial\omega}\right\}$$

In case nonlinear distortion of the HF pulse is negligible, the step with the Burgers Eqn. is neglected

3.2 Discretization of transversal derivative

The diffraction of the HF pulse is determined by the transversal derivative

$$\nabla_\perp^2\widehat{p}\left(z,\underline{r}_\perp,\omega\right)=\frac{\partial^2\widehat{p}\left(z,\underline{r}_\perp,\omega\right)}{\partial x^2}+\frac{\partial^2\widehat{p}\left(z,\underline{r}_\perp,\omega\right)}{\partial y^2}$$

For the location of $\underline{r}_{mn}=\left(x_m,y_n\right)$ we approximate the derivative as

$$\frac{\partial^2\widehat{p}\left(z,x_m,y_n,\omega\right)}{\partial x^2}+\frac{\partial^2\widehat{p}\left(z,x_m,y_n,\omega\right)}{\partial x^2}=\sum_{l=-L}^{L}a_l\left\{\widehat{p}\left(x_m-l\Delta,y_n,\ldots\right)+\widehat{p}\left(x_m,y_n-l\Delta,\ldots\right)\right\}$$

$$\Delta=\frac{2\pi}{k_s}\qquad k_s=\frac{2\pi}{\Delta}$$

where Δ is the sampling interval in the x-y directions, and k_s is the transversal spatial sampling frequency. The coefficients a_l can be determined through the following Fourier analysis.

$$F_x\left\{\sum_{l=-L}^{L} a_l \hat{p}\left(x_m - l\Delta, y_n, \ldots\right)\right\} = \hat{p}\left(k_x, y_n, \ldots\right)\sum_{l=-L}^{L} a_l e^{il2\pi k_x/k_s}$$

$$k_x \in \left[-k_s/2, k_s/2\right]$$

The coefficinets are then determined for best approximation of the Fourier polynimial to

$$-k_x^2 \approx \sum_{l=-L}^{L} a_l e^{il2\pi k_x/k_s}$$

This can be done by minimization of the following functional

$$J = \int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)\left|k_x^2 + \sum_{l=-L}^{L} a_l e^{il2\pi k_x/k_s}\right|^2$$

$$= \int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)\left(k_x^2 + \sum_{l=-L}^{L} a_l e^{il2\pi k_x/k_s}\right)\left(k_x^2 + \sum_{l=-L}^{L} a_l e^{il2\pi k_x/k_s}\right)^*$$

with respect to a_l. $W\left(k_x\right)$ is a window weighting function.

$$\frac{\partial J}{\partial a_q} = 2\int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)\left(k_x^2 + \sum_{l=-L}^{L} a_l e^{il2\pi k_x/k_s}\right)e^{-iq2\pi k_x/k_s} = 0$$

which gives

$$\sum_{l=-L}^{L} a_l \int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)e^{-i(q-l)2\pi k_x/k_s} = -\int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)k_x^2 e^{-iq2\pi k_x/k_s}$$

This can be formulated as the linear set of equations

$$\sum_{l=-L}^{L} A_{ql} a_l = B_q \qquad q = -L, \ldots, L$$

$$A_{ql} = \int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)e^{-i(q-l)2\pi k_x/k_s} \qquad B_q = -\int_{-k_s/2}^{k_s/2} dk_x W\left(k_x\right)k_x^2 e^{-iq2\pi k_x/k_s}$$

For $W\left(k_x\right) = 1$ we get the standard Fourier coefficients

$$A_{ql} = \int_{-k_s/2}^{k_s/2} dk_x e^{i(q+l)2\pi k_x/k_s} = k_s \delta_{ql} \qquad \text{which gives}$$

$$a_l = -\frac{1}{k_s}\int_{-k_s/2}^{k_s/2} dk_x k_x^2 e^{il2\pi k_x/k_s}$$

However, this can give some annoying oscillations in the Fourier polynomial close to the ends of the interval $(\pm k_s/2)$, and therefore one can get nicer approximation of the polynomial towards the edges. A typical weighting function can be a Hamming/Hanning window (http://en.wikipedia.org/wiki/Window_function)

$$W(k_x) = 0.54 - 0.46\cos\frac{2\pi k_x}{k_s} \qquad \text{Hamming window}$$

$$W(k_x) = 0.5 - 0.5\cos\frac{2\pi k_x}{k_s} \qquad \text{Hanning}$$

As k_x^2 is even around zero, the coefficients a_l will be even in l, i.e $a_{-l} = a_l$. Eq.(XX) can hence be reduced in dimension. Provided $W(k_x)$ is even in k_x we get

$$A_{ql} = \int_{-k_s/2}^{k_s/2} dk_x W(k_x)\cos\{(q-l)2\pi k_x/k_s\} = A_{-q,-l} \qquad B_q = -\int_{-k_s/2}^{k_s/2} dk_x W(k_x) k_x^2 \cos\{q2\pi k_x/k_s\} = B_{-q}$$

The last set of Eqs (-L,…,-1) is hence equal to the set of Eqs, (1,…L), and we can write

$$\sum_{l=0}^{L} C_{ql} a_l = B_q \qquad q = 0,\ldots,L$$

$$C_{ql} = \int_{-k_s/2}^{k_s/2} dk_x W(k_x)\left\{\cos\left((q-l)2\pi k_x/k_s\right) + \cos\left((q+l)2\pi k_x/k_s\right)\right\} \qquad \text{for } l \neq 0$$

$$C_{q0} = \int_{-k_s/2}^{k_s/2} dk_x W(k_x)\cos\left(q2\pi k_x/k_s\right)$$

The linear part of the operator splitting, Eq.(XX) now takes the form

$$\hat{p}\left(z_i,\underline{r}_\perp,\omega\right)=\hat{p}\left(z_{i-1},\underline{r}_\perp,\omega\right)\exp\left\{-i\frac{c_0}{2\omega}\int_{z_{i-1}}^{z_i}ds\sum_{l=-L}^{L}a_l\left\{\overline{p}\left(s,x_m-l\Delta,y_n,\omega\right)+\overline{p}\left(s,x_m,y_n-l\Delta,\omega\right)\right\}/\overline{p}\left(s,\underline{r}_{mn},\omega\right)\right\}$$

$$\times\exp\left\{-i\frac{\omega}{2c_0}\int_{z_{i-1}}^{z_i}dsH\left(s,\underline{r}_\perp,\omega\right)-i\omega\delta\tau_n(z_i,\underline{r}_\perp)\right\}$$

$$\times\exp\left\{ik_\omega\int_{z_{i-1}}^{z_i}ds\;\alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L\left(s,\underline{r}_\perp\right)\left(\frac{\overline{p}\left(s,\underline{r}_\perp,\omega-\omega_L\right)+\overline{p}\left(s,\underline{r}_\perp,\omega+\omega_L\right)}{2\overline{p}\left(s,\underline{r}_\perp,\omega\right)}-1\right)\right\}$$

$$\times\exp\left\{-k_\omega\int_{z_{i-1}}^{z_i}ds\;\alpha\left(s,\underline{r}_\perp\right)\sin\omega_L\tau_L\left(s,\underline{r}_\perp\right)\frac{\overline{p}\left(s,\underline{r}_\perp,\omega-\omega_L\right)-\overline{p}\left(s,\underline{r}_\perp,\omega+\omega_L\right)}{2\overline{p}\left(s,\underline{r}_\perp,\omega\right)}\right\}$$

$$\delta\tau_n(z_i,\underline{r}_\perp)=-\frac{1}{c_0}\int_{z_{i-1}}^{z_i}ds\;\alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)\qquad\tau_n(z_i,\underline{r}_\perp)=-\frac{1}{c_0}\int_0^{z_i}ds\;\alpha\left(s,\underline{r}_\perp\right)\cos\omega_L\tau_L(s,\underline{r}_\perp)$$

4. Suppression of Class I/II reverberations

4.1 Model of Class I/II reverberations

According to [XX] we get the sum of the Class I and Class II reverberation noise as

$$dW_i\left(\omega;t_1,t_2\right)=dt_1dt_2U_{ri}\left(\omega;t_1,t_2\right)\left\{\underbrace{Q_i\left(\omega;t_1,t_2\right)V_k\left(\omega;t_1;p_k\right)e^{-i\omega p_k\tau(t_1)}}_{Class\ I}+\underbrace{V_k\left(\omega;t_i-t_1+t_2;p_k\right)e^{-i\omega p_k\tau(t_i-t_1+t_2)}}_{Class\ II}\right\}$$

where t_1 is the time lag to the 1st scatterer for Class I reverberations, t_2 is the time lag to the 2nd scatterer, and $t_3=t_i-(t_1-t_2)$ is the time lag to the 3rd scatterer for Class I reverberations. For Class II reverberations, the 1st and 3rd scatterer inter changes. The subscript i denotes the depth time interval T_i around t_i, $U_{ri}\left(\omega;t_1,t_2\right)$ is the Class II pulse reverberation noise with the 1st scatterer at $t_3=t_i-(t_1-t_2)$, $Q_i\left(\omega;t_1,t_2\right)$ takes care of the difference between Class I/II reverberations due to differences in the transmit/receive beams for zero LF pulse. $V_k\left(\omega;t_1;p_k\right)$ represents the nonlinear distortion for Class I reverberations where the 1st scatterer is at t_1, while and $V_k\left(\omega;t_i-t_1+t_2;p_k\right)$ represents the nonlinear distortion for Class I reverberations where the 1st scatterer is at $t_3=t_i-(t_1-t_2)$. The subscript k represents the transmit pulse number with p_k as the transmitted LF pulse amplitude. $p_k\tau(t_1)$ is the nonlinear propagation delay for Class I reverberations, while $p_k\tau(t_i-t_1+t_2)$ is the nonlinear propagation delay for Class II reverberations.

At the 1st scattering, the amplitude of the LF pulse drops so much that the nonlinear propagation lag and pulse form distortion can be neglected after the 1st scattering that has the location t_1 for Class I and $t_3=t-t_1+t_2$ for Class II. As the propagation distance to the 1st scatterer is highest for Class II, the nonlinear propagation lag and pulse form distortion is highest for Class II reverberations.

The same situation is also found for the nonlinear self distortion of the HF pulse, where one can neglect the nonlinear self distortion after the 1st scattering due to the drop in amplitude at the scattering. The nonlinear self distortion is hence highest for the Class II reverberations as the propagation distance to the 1st scattering is highest. The nonlinear self distortion is found as an added attenuation of the fundamental band of the HF pulse, and hence increases the amplitude of Q in the order of ~ 1 – 2 dB from that found with pure linear propagation.

From [XX] we see that we can obtain the linear $Q_i\left(\omega;t_1,t_2\right)$ from beam simulations as

$$Q\left(\omega;\underline{r}_1,\underline{r}_3\right)=\frac{dY_{3I}\left(\omega;\underline{r}_1,\underline{r}_3\right)}{dY_{3II}\left(\omega;\underline{r}_1,\underline{r}_3\right)}=\frac{H_r\left(\underline{r}_3,\omega;\underline{r}_r\right)H_t\left(\underline{r}_1,\omega;\underline{r}_t\right)}{H_r\left(\underline{r}_1,\omega;\underline{r}_r\right)H_t\left(\underline{r}_3,\omega;\underline{r}_t\right)}$$

where $H_t\left(\underline{r}_1,\omega;\underline{r}_t\right)$ is the spatial frequency response at $\underline{r}_1$ of the transmit beam with focus at $\underline{r}_t$. The added attenuation due to nonlinear self distortion must be obtained from a nonlinear simulation of both $H_t\left(\underline{r}_1,\omega;\underline{r}_t\right)$ and $H_t\left(\underline{r}_3,\omega;\underline{r}_t\right)$, and will depend on the transmitted HF amplitude. $H_r\left(\underline{r}_1,\omega;\underline{r}_r\right)$ is the spatial frequency response at $\underline{r}_1$ of the receive beam with focus at $\underline{r}_t$.

For the nonlinear pulse form distortion we have the following relations

$$V_k\left(\omega;t_1;p_k\right)=\frac{p\left(z_1,\underline{0},\omega;p_k\right)}{p\left(z_1,\underline{0},\omega;0\right)}=\frac{p_+\left(z_1,\omega\right)}{p_0\left(z_1,\omega\right)}$$

$$V_k\left(\omega;t_i-t_1+t_2;p_k\right)=\frac{p\left(z_i-z_1+z_2,\underline{0},\omega;p_k\right)}{p\left(z_i-z_1+z_2,\underline{0},\omega;0\right)}=\frac{p_+\left(z_i-z_1+z_2,\omega\right)}{p_0\left(z_i-z_1+z_2,\omega\right)}$$

where $z_n=t_n/c_0$, $n=1,2,i$ and the *p(...)* are the HF pulses as obtained from simulation of Eqs.(XX,XX). The nonlinear propagation delay is also $p_k\tau(t_n)$ is also obtained from simulations of Eqs.(XX,XX). Class II reverberations have the longest propagation before the 1st scattering, and hence has the largest nonlinear self-distortion of the forward propagating pulse.

The 3rd order reverberation noise with the 1st scatterer at t_1 and the 2nd scatterer at t_2, can then be written as

$$dW_{ki}\left(\omega;t_1,t_2\right)=dt_1dt_2U_{ri}\left(\omega;t_1,t_2\right)A_{ki}\left(\omega;t_1,t_2\right)e^{-i\omega\left[\tau_k\left(t_i-t_1+t_2\right)+\tau_k\left(t_1\right)\right]/2}$$

where

$$A_{ki}(\omega;t_1,t_2) = |A_{ki}(\omega;t_1,t_2)| e^{-i\varphi_{Ak}(\omega;t_1,t_2)}$$

$$= Q_i(\omega;t_1,t_2) V_k(\omega;t_1) e^{i\omega[\tau_k(t_i-t_1+t_2)-\tau_k(t_1)]/2} + V_k(\omega;t_i-t_1+t_2) e^{-i\omega[\tau_k(t_i-t_1+t_2)-\tau_k(t_1)]/2}$$

$$= \left[Q_i(\omega;t_1,t_2)V_k(\omega;t_1) + V_k(\omega;t_i-t_1+t_2)\right]\cos\left(\omega\left[\tau_k(t_i-t_1+t_2)-\tau_k(t_1)\right]/2\right)$$

$$+i\left[Q_i(\omega;t_1,t_2)V_k(\omega;t_1) - V_k(\omega;t_i-t_1+t_2)\right]\sin\left(\omega\left[\tau(t_i-t_1+t_2)-\tau_k(t_1)\right]/2\right)$$

The total noise for a variability of scatterers is then

$$W_{ki}(\omega) = \int_0^{t_{2m}} dt_2 \int_{t_2}^{(t_i+t_{2m})/2} dt_1 U_{ri}(\omega;t_1,t_2) A_{ki}(\omega;t_1,t_2) e^{-i\omega[\tau_k(t_i-t_1+t_2)+\tau_k(t_1)]/2}$$

$$= \int_0^{(t_i+t_{2m})/2} dt_1 \int_0^{\min(t_1,t_{2m})} dt_2 U_{ri}(\omega;t_1,t_2) A_{ki}(\omega;t_1,t_2) e^{-i\omega[\tau_k(t_i-t_1+t_2)+\tau_k(t_1)]/2}$$

where for $t_2 > t_{2m}$ we get Class III reverberations. We extract a delay for correction, and modifies this equation to

$$W_{ki}(\omega) = e^{-i\omega\tau_{nki}} N_{ki}(\omega)$$

$$N_{ki}(\omega) = \int_0^{t_{2m}} dt_2 \int_{t_2}^{(t_i+t_{2m})/2} dt_1 U_{ri}(\omega;t_1,t_2) A_{ki}(\omega;t_1,t_2) e^{-i\omega[\tau_k(t_i-t_1+t_2)+\tau_k(t_1)-2\tau_{nki}]/2}$$

The total received signal in T_i can be written as

$$Y_{ki}(\omega;p_k) = e^{-i\omega\tau_{ki}}\left\{S_{li}(\omega;p_k) + S_{ni}(\omega;p_k)\right\} + e^{-i\omega p_k \tau_{nki}} N_{ki}(\omega;p_k)$$

where S_{li} is the linearly scattered signal and S_{ni} is the nonlinearly scattered signal that is given with the general nonlinear dependency on p_k that includes nonlinear scattering from micro bubbles. For fluids and tissue we generally can approximate $S_{ni}(\omega;p_k) = p_k S_{ni}(\omega)$.
We note that the delay component can be written as

$$\tau_{nki}(t_i;t_1,t_2) = \left[\tau_k(t_i-t_1+t_2) + \tau_k(t_1)\right]/2 + \tau_{Ak}(t_i;t_1,t_2)$$

where $\tau_{Ak}(t_i;t_1,t_2)$ is a potential component of $\varphi_{Ak}(\omega;t_1,t_2)$ that is linear in frequency. If this is zero or negligible, the nonlinear propagation delay for the combined Class I/II reverberation is the mean-average of the nonlinear propagation delay of Class I and Class II reverberations.

When the transducer array is the major 2nd scatterer/reflector, we only get contribution to the integral in Eq.(XX) for $t_2 = 0$, and the reverberation noise takes the form

$$N_{ki}(\omega) = \int_0^{t_i/2} dt_1 U_{ri}(\omega;t_1,t_2) A_{ki}(\omega;t_1,t_2) e^{-i\omega[\tau_k(t_i-t_1)+\tau_k(t_1)-2\tau_{nki}]/2}$$

If further the nonlinear propagation delay increases linearly with depth, i.e. $\tau_k(t) = a_k t$, we get

$$[\tau_k(t_i - t_1) + \tau_k(t_1)]/2 = a_k t_i/2 = \tau_k(t_i/2)$$

4.2 Suppression of Class I/II reverberations with 2-level pulses

A. Using only measurements

For each beam direction we transmit two pulse complexes with opposite polarity of the LF pulse. The nonlinear scattering is weak or located to points so that it is neglected. We hence get the received signals

$$Y_{+i}(\omega) = e^{i\omega\tau_i} S_{+i}(\omega) + e^{i\omega\tau_{ni}} N_{+i}(\omega)$$

$$Y_{-i}(\omega) = e^{-i\omega\tau_i} S_{-i}(\omega) + e^{-i\omega\tau_{ni}} N_{-i}(\omega)$$

There exists a correction filter $H_{ci}(\omega)$ so that

$$H_{ci}(\omega) N_{-i}(\omega) = N_{+i}(\omega)$$

and a correction delay $\tau_{ci} = \tau_{ni}$ so that the noise term can be highly suppressed, and we get a noise suppressed imaging signal from the time interval T_i as

$$Z_i(\omega) = e^{-i\omega\tau_{ni}} Y_{+i}(\omega) - e^{i\omega\tau_{ni}} H_{ci}(\omega) Y_{-i}(\omega) = e^{i\omega(\tau_i-\tau_{ni})} S_{+i}(\omega) - e^{-i\omega(\tau_i-\tau_{ni})} H_{ci}(\omega) S_{-i}(\omega)$$

If we transmit a 3-pulse sequence with the LF amplitudes $p_1 = +p_{LF}$, $p_0 = 0$, $p_3 = -p_{LF}$ designed by the (+/0/-) subscript we can also separate out a non-distorted 1st order scattering signal as discussed in Section 4.XX.

The challenge is however how to estimate the correction delay and filter.

If the signal-to-noise ratio is adequately low, we can estimate the nonlinear propagation delay and pulse form distortion filter from the measured signals as

$$\tau(t_1) = \frac{1}{2} delay\{y_+(t), y_-(t)\} \qquad V_1(\omega;t_1) = \frac{e^{-i\omega\tau(t_1)} Y_+(\omega;t_1)}{e^{i\omega\tau(t_1)} Y_-(\omega;t_1) + N}$$

$$\tau_c(t_i) = \angle\left\{\int_0^{t_i/2} dt_1 \tilde{y}_+(t_1)\tilde{y}_+(t_i - t_1)A_i(\omega;t_1,t_i - t_1)\right\}$$

$$H_{ci}(\omega;t_i) = \frac{e^{-i\omega\tau_c(t_i)}\int_0^{t_i/2} dt_1 \tilde{y}_+(t_1)\tilde{y}_+(t_i - t_1)A_i(\omega;t_1,t_i - t_1)}{\int_0^{t_i/2} dt_1 \tilde{y}_-(t_1)\tilde{y}_-(t_i - t_1)\left\{Q_i(\omega;t_1)+1\right\}}$$

$$A_i(\omega;t_1,t_i - t_1) = Q_i(\omega;t_1)V_1(\omega;t_1)e^{-i\omega 2\tau(t_1)} + V_1(\omega;t_i - t_1)e^{-i\omega 2\tau(t_i - t_1)}$$

where $\tilde{y}_+(t)$ is the envelope of the received signal, compensated for absorption. For adequately linear variation of the nonlinear propagation delay, we have from Eq.(XX) that

$$\tau_c(t_i) \approx 2\tau(t_i/2)$$

As there are inaccuracies on the compensation for absorption, it might be better to use this approximate correction delay, compared to that from Eq.(XXb). If the there are other strong 2nd scatterers than the transducer array, the expressions modifies as

$$\tau_c(t_i) = \angle\left\{\int_0^{t_i/2} dt_1 \tilde{y}_+(t_1)\int_0^{t_1} dt_2 \tilde{y}_+(t_2)\tilde{y}_+(t_i - t_1 + t_2)A_i(\omega;t_1,t_i - t_1 + t_2)\right\}$$

$$H_{ci}(\omega;t_i) = \frac{e^{-i\omega\tau_c(t_i)}\int_0^{t_i/2} dt_1 \tilde{y}_+(t_1)\int_0^{t_1} dt_2 \tilde{y}_+(t_2)\tilde{y}_+(t_i - t_1 + t_2)A_i(\omega;t_1,t_i - t_1 + t_2)}{\int_0^{t_i/2} dt_1 \tilde{y}_-(t_1)\int_0^{t_1} dt_2 \tilde{y}_-(t_2)\tilde{y}_-(t_i - t_1 + t_2)\left\{Q_i(\omega;t_1)+1\right\}}$$

$$A_i(\omega;t_1,t_i - t_1 + t_2) = Q_i(\omega;t_1,t_2)V_1(\omega;t_1)e^{-i\omega 2\tau(t_1)} + V_1(\omega;t_i - t_1 + t_2)e^{-i\omega 2\tau(t_i - t_1 + t_2)}$$

B. Supporting simulations of nonlinear propagation delay and pulse form distortion

With low signal-to-noise ratios the estimation of the nonlinear propagation delay and pulse form distortion in Eq.(XXa) can have so much error that we get poor results of the procedure under 4.2A. In this case it can be advantegous to simulate at least one of the nonlinear propagation delay and pulse form distortion according to Section 2.XX.

We then assume a material parameter vector

$$\underline{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_N) = \left\{\alpha(t_1),\alpha(t_2),\ldots,\alpha(t_N)\right\}$$

that allows us to simulate

$$V_1\left(\omega;t_1;\underline{\alpha}\right)e^{i\omega 2\tau(t_1;\underline{\alpha})}=\frac{p_+\left(z_1,\omega;\underline{\alpha}\right)}{p_-\left(z_1,\omega;\underline{\alpha}\right)}$$

and then carry through the calculations as above

$$A_i\left(\omega;t_1,t_i-t_1+t_2v\right)=Q_i\left(\omega;t_1,t_2\right)V_1\left(\omega;t_1;\underline{\alpha}\right)e^{-i\omega 2\tau(t_1;\underline{\alpha})}+V_1\left(\omega;t_i-t_1+t_2;\underline{\alpha}\right)e^{-i\omega 2\tau(t_i-t_1+t_2;\underline{\alpha})}$$

$$\tau_c(t_i;\underline{\alpha})=\angle\left\{\int_0^{t_i/2}dt_1\tilde{y}_+\left(t_1\right)\int_0^{t_1}dt_2\tilde{y}_+\left(t_2\right)\tilde{y}_+\left(t_i-t_1+t_2\right)A_i\left(\omega;t_1,t_i-t_1+t_2;\underline{\alpha}\right)\right\}$$

$$H_{ci}\left(\omega;t_i;\underline{\alpha}\right)=\frac{e^{-i\omega\tau_c(t_i)}\int_0^{t_i/2}dt_1\tilde{y}_+\left(t_1\right)\int_0^{t_1}dt_2\tilde{y}_+\left(t_2\right)\tilde{y}_+\left(t_i-t_1+t_2\right)A_i\left(\omega;t_1,t_i-t_1+t_2;\underline{\alpha}\right)}{\int_0^{t_i/2}dt_1\tilde{y}_-\left(t_1\right)\int_0^{t_1}dt_2\tilde{y}_-\left(t_2\right)\tilde{y}_-\left(t_i-t_1+t_2\right)\left\{Q_i\left(\omega;t_1\right)+1\right\}}$$

The parameter vector $\underline{\alpha}$ must then be adjusted so that the noise is minimized.

4.3 Suppression of Class I/II reverberations with 3-level pulses

A. XXX

Through simulation of Eqs.(XX,XX) for $p_1=+p_{LF}$, $p_0=0$, $p_3=-p_{LF}$ and define (+/0/- subscript indicates transmission with $+p_{LF},0,-p_{LF}$ respectively

$$V_{+i}\left(\omega;t_1\right)=\frac{p_+\left(z_1,\omega\right)}{p_0\left(z_1,\omega\right)}\qquad V_{-i}\left(\omega;t_1\right)=\frac{p_-\left(z_1,\omega\right)}{p_0\left(z_1,\omega\right)}$$

$$V_{+i}\left(\omega;t_i-t_1+t_2\right)=\frac{p_+\left(z_i-z_1+z_2,\omega\right)}{p_0\left(z_i-z_1+z_2,\omega\right)}\qquad V_{-i}\left(\omega;t_i-t_1+t_2\right)=\frac{p_-\left(z_i-z_1+z_2,\omega\right)}{p_0\left(z_i-z_1+z_2,\omega\right)}$$

$Q_i\left(\omega;t_1,t_2\right)$ can be obtained from simulation of Eq.(XX) where nonlinear simulation of the transmit beams is required to

must be estimated from a simulation of combined transmit and receive beams with defined we are together with the nonlinear propagation delay able to determine $A_{ki}\left(\omega;t_1,t_2\right)$

$$A_{+i}\left(\omega;t_1,t_2;p_{LF}\right)=\left|A_{-+i}\left(\omega;t_1,t_2;p_{LF}\right)\right|e^{-i\varphi_A\left(\omega;t_1,t_2;p_{LF}\right)}$$

$$=Q_i\left(\omega;t_1,t_2\right)V_{+i}\left(\omega;t_1;p_{LF}\right)e^{i\omega p_{LF}\left[\tau(t_i-t_1+t_2)-\tau(t_1)\right]/2}+V_{ki}\left(\omega;t_i-t_1+t_2;p_{LF}\right)e^{-i\omega p_{LF}\left[\tau(t_i-t_1+t_2)-\tau(t_1)\right]/2}$$

$$A_{-i}\left(\omega;t_1,t_2;-p_{LF}\right)=\left|A_{-i}\left(\omega;t_1,t_2;-p_{LF}\right)\right|e^{-i\varphi_A\left(\omega;t_1,t_2;-p_{LF}\right)}$$

$$=Q_i\left(\omega;t_1,t_2\right)V_{-i}\left(\omega;t_1;-p_{LF}\right)e^{-i\omega p_{LF}\left[\tau(t_i-t_1+t_2)-\tau(t_1)\right]/2}+V_{-i}\left(\omega;t_i-t_1+t_2;-p_{LF}\right)e^{+i\omega p_{LF}\left[\tau(t_i-t_1+t_2)-\tau(t_1)\right]/2}$$

$$A_{0i}\left(\omega;t_1,t_2;0\right)=\left|A_{0i}\left(\omega;t_1,t_2;0\right)\right|e^{-i\varphi_A\left(\omega;t_1,t_2;0\right)}=Q_i\left(\omega;t_1,t_2\right)+1$$

The nonlinear propagation delays are obtained in the same simulation as

$$\tau_+(t_i;t_1,t_2)=p_{LF}\tau_W(t_i;t_1,t_2)=p_{LF}\left[\tau(t_i-t_1+t_2)+\tau(t_1)\right]/2$$

$$\tau_-(t_i;t_1,t_2)=-p_{LF}\tau_W(t_i;t_1,t_2)=-p_{LF}\left[\tau(t_i-t_1+t_2)+\tau(t_1)\right]/2$$

The total noise for a variability of scatterers is then

$$W_{ki}\left(\omega;p_k\right)=\int_0^{t_{2m}}dt_2\int_{t_2}^{(t_i+t_{2m})/2}dt_1U_{ri}\left(\omega;t_1,t_2\right)A_{ki}\left(\omega;t_1,t_2;p_k\right)e^{-i\omega p_k\left[\tau(t_i-t_1+t_2)+\tau(t_1)\right]/2}$$

$$=\int_0^{(t_i+t_{2m})/2}dt_1\int_0^{\min\left(t_1,t_{2m}\right)}dt_2U_{ri}\left(\omega;t_1,t_2\right)A_{ki}\left(\omega;t_1,t_2;p_k\right)e^{-i\omega p_k\left[\tau(t_i-t_1+t_2)+\tau(t_1)\right]/2}$$

where for $t_2 > t_{2m}$ we get Class III reverberations. We extract a delay for correction, and modifies this equation to

$$W_{ki}\left(\omega;p_k\right)=e^{-i\omega p_k\tau_{nki}}N_{ki}\left(\omega;p_k\right)$$

$$N_{ki}\left(\omega;p_k\right)=\int_0^{t_{2m}}dt_2\int_{t_2}^{(t_i+t_{2m})/2}dt_1U_{ri}\left(\omega;t_1,t_2\right)A_{ki}\left(\omega;t_1,t_2;p_k\right)e^{-i\omega p_k\left[\tau(t_i-t_1+t_2)+\tau(t_1)-2\tau_{nki}\right]/2}$$

and the total received signal in T_i can be written as

$$Y_{ki}\left(\omega;p_k\right)=e^{-i\omega p_k\tau_i}\left\{S_{li}\left(\omega;p_k\right)+S_{ni}\left(\omega;p_k\right)\right\}+e^{-i\omega p_k\tau_{nki}}N_{ki}\left(\omega;p_k\right)$$

where S_{li} is the linearly scattered signal and S_{ni} is the nonlinearly scattered signal that is given with the general nonlinear dependency on p_k that includes nonlinear scattering from micro bubbles. From fluids and tissue we generally can approximate $S_{ni}\left(\omega;p_k\right)=p_kS_{ni}\left(\omega\right)$.

5. Combined suppression of Class I/II reverberations

5.1 Transducer array main reflector

When the transducer array is the main reflector we mainly get contribution for $t_2 = 0$. In this case we get

$$N_{ki}\left(\omega; p_k\right) = \int_0^{t_i/2} dt_1 U_{ri}\left(\omega; t_1\right) A_{ki}\left(\omega; t_1; p_k\right) e^{-i\omega p_k\left[\tau(t_i - t_1) + \tau(t_1) - 2\tau_{nki}\right]/2}$$